

SIMPLE TYPE IS NOT A BOUNDARY PHENOMENON

by

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Abstract

This is an expository article, explaining recent work by D. Groisser and myself [GS] on the extent to which the boundary region of moduli space contributes to the “simple type” condition of Donaldson theory. The presentation is intended to complement [GS], presenting the essential ideas rather than the analytical details. It is shown that the boundary region of moduli space contributes $6/64$ of the homology required for simple type, regardless of the topology or geometry of the underlying 4-manifold. The simple type condition thus reduces to a statement about the interior of moduli space, namely that the interior of the $k + 1$ st ASD moduli space, intersected with two representatives of (4 times) the point class, be homologous to 58 copies of the k -th moduli space. This is peculiar, since the only known embeddings of the k -th moduli space into the $k + 1$ st involve Taubes patching, and the image of such an embedding lies entirely in the boundary region.

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In this paper I discuss some recent work of David Groisser and myself [GS] on how the “simple type” condition of Donaldson theory is related to the geometry of the moduli spaces of anti-self-dual connections over a given 4-manifold. But before I begin, I must answer the obvious nagging question: *Haven’t the Seiberg-Witten equations made all of Donaldson theory obsolete?* I obviously think not. While SW theory has eclipsed much of Donaldson theory, it has actually made other uses of Donaldson invariants *more* practical. This paper, I hope, will serve as an example of how to extract insight from Donaldson theory in the post-Seiberg-Witten era.

A Historical Digression

Once upon a time, the Yang-Mills (YM) equations were proposed and were studied without regard to topological consequences. The motivation was from physics: the equations accurately describe physics at the subnuclear scale, and the “standard model” is an $SU(3) \times SU(2) \times U(1)$ gauge theory. In the 1970s and 1980s, people began to write down solutions to the YM equations. Of particular interest were anti-self-dual (ASD) connections, which automatically satisfy the YM equations. People began to look at “moduli spaces” of ASD connections, both over manifolds of physical interest (\mathbb{R}^4 , or equivalently S^4) and over more general 4-manifolds. In addition to aiding our understanding of the connections themselves, these moduli spaces were of interest in their own right, especially over complex manifolds, where ASD connections correspond to holomorphic vector bundles.

Donaldson’s brilliant insight was that the topology of moduli spaces (and in particular how they sit in the larger space of all connections modulo gauge transformations) could tell a lot about the differential topology of the underlying 4-manifolds. Donaldson invariants capture the essential topological information about the moduli spaces, and proved very useful for classifying smooth 4-manifolds. In fact, they were *so* successful that the previous quest, to understand the YM equations and the ASD moduli spaces for their own sake, was largely neglected.

By 1990, then, mathematicians primarily studied the YM equations in order to understand moduli spaces, studied moduli spaces in order to understand Donaldson invariants, and studied Donaldson invariants in order to aid in the classification of smooth 4-manifolds. This is schematically shown in Figure 1. This was a grand and difficult project, with a huge number of practitioners making gradual progress.

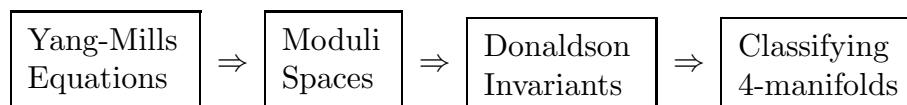


Figure 1. The Traditional Flow of Ideas

The Seiberg-Witten Revolution

In late 1994, this project was largely made irrelevant by the advent of the Seiberg-Witten equations [W]. The SW invariants are far easier to compute than Donaldson invariants, and are generally believed to carry exactly the same information. Witten’s formula

[W] for the Donaldson invariants in terms of the SW invariants is almost universally believed, although as of this writing a mathematical proof is still lacking. As far as classifying smooth 4-manifolds goes, just about anything that can be done with Donaldson theory can be done, far more easily, with SW invariants.

The last arrow in Figure 1 must therefore be abandoned. Indeed, the middle arrow is also largely superceded, as (modulo a proof of the Witten conjecture) the Donaldson invariants are most easily computed by first computing the SW invariants and then applying Witten’s formula. The flow of ideas in Figure 1 is simply obsolete.

$SU(2)$ gauge theory, however, is *not* obsolete, if we remember the original interest in the YM equations and the moduli spaces. We just have to reverse the arrows in Figure 1!

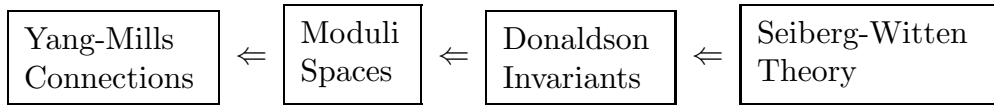


Figure 2. The New Paradigm

We can use SW theory to gain insight into the structure of Donaldson invariants. We can use Donaldson invariants to tell us about moduli spaces. Finally, we can use moduli spaces to tell us about solutions to the YM equations. Perhaps this is not a “politically correct” program; nothing points to a classification of 4-manifolds! But the structure of moduli spaces can be extremely interesting, so let’s get to work using our new-found tools.

Our Results

This paper is an exercise along the lines of Figure 2. I will take as given that a manifold has simple type (as all smooth orientable 4-manifolds with $b_+ > 1$ are believed to have), and see what that says about the structure of its moduli spaces. The results are quite surprising!

Simple type says that the $k+1$ st moduli space \mathcal{M}_{k+1} , intersected with certain varieties, has the homology of 64 copies of the k -th moduli space \mathcal{M}_k . I will show that the portion of (a small perturbation of) \mathcal{M}_{k+1} near the boundary, cut down, looks like exactly 6 copies of \mathcal{M}_k , regardless of the topology or geometry of the underlying 4-manifold. Simple type thus implies that the *interior* of \mathcal{M}_{k+1} , intersected with certain varieties, has the homology of 58 copies of \mathcal{M}_k . This is quite surprising, since the only known embeddings of \mathcal{M}_k into \mathcal{M}_{k+1} involve Taubes patching, and have images near the boundary of \mathcal{M}_{k+1} . Nobody has the slightest idea of what \mathcal{M}_k has to do with the interior of \mathcal{M}_{k+1} , yet for all known 4-manifolds with $b_+ > 1$, they appear to be closely related. This is a mystery that warrants further investigation.

Here is an outline for the rest of the paper. First I review the definitions of the Donaldson invariants, and of simple type, and state the result precisely. Then I sketch the proof, which has three essential ingredients. First there is a choice of the geometric representative of the point class. Then there is an approximate formula for the curvature of a connection in the boundary region of \mathcal{M}_{k+1} . Finally there is a very naive calculation that illustrates clearly why the boundary region of \mathcal{M}_{k+1} contributes exactly 6 copies of

\mathcal{M}_k , rather than any other number. It takes quite a bit of analysis to thoroughly justify the approximate formula and the naive calculation; all that can be found in [GS]. Here I will concentrate on presenting the essential ideas as clearly as possible, making simplifying assumptions, as needed, along the way.

What Is Simple Type, Anyway?

Let X be an oriented 4-manifold, let $G = SU(2)$ or $SO(3)$ and let \mathcal{B}_k^* be the space of irreducible connections (up to gauge equivalence) on P_k , the principal G bundle of instanton number k over X . Let $\mathcal{M}_k \subset \mathcal{B}_k^*$ be the space of irreducible connections on P_k with anti-self-dual curvature, modulo gauge transformations.

Donaldson [D1, D2] defined a map $\mu : H_i(X, \mathbb{Q}) \rightarrow H^{4-i}(\mathcal{B}_k^*, \mathbb{Q})$, $i = 0, 1, 2, 3$, whose image freely generates the rational cohomology of \mathcal{B}_k^* . Donaldson invariants are defined by pairing the fundamental class of \mathcal{M}_k with products of μ of the homology classes of X , where k is chosen so that the dimensions match. Formally, for elements $[\Sigma_1], \dots, [\Sigma_n] \in H_*(X)$, we write

$$D([\Sigma_1] \dots [\Sigma_n]) = \mu([\Sigma_1]) \smile \dots \smile \mu([\Sigma_n])[\mathcal{M}_k]. \quad (1)$$

Now let x be the point class in $H_0(X)$, and let ω be any formal product of classes in $H_*(X)$. The simple type condition is that, for all ω ,

$$D(x^2\omega) = 4D(\omega). \quad (2)$$

Of course, the “fundamental class of \mathcal{M}_k ” is usually not well defined, as \mathcal{M}_k is typically not compact. The usual way to make sense of (1) and (2) is with geometric representatives. One finds finite-codimension varieties V_Σ in \mathcal{B}_k^* that are Poincare dual to $\mu([\Sigma])$. One then counts points, with sign, in $V_{\Sigma_1} \cap \dots \cap V_{\Sigma_n} \cap \mathcal{M}_k$. To make a topological invariant one must show that the number of intersection points is independent of auxiliary data, such as the metric and the choice of representatives. This requires careful analysis of the bubbling-off phenomena that make \mathcal{M}_k noncompact.

Unfortunately, $\mu(x)$ is not an integral class in $H^4(\mathcal{B}^*)$. However, $-4\mu(x)$ is an integral class. Let ν_1 and ν_2 be two (generic) geometric representatives of $-4\mu(x)$. The simple type condition can be rewritten as

$$\#(\mathcal{M}_{k+1} \cap \nu_1 \cap \nu_2 \cap V_\omega) = 64\#(\mathcal{M}_k \cap V_\omega), \quad (3)$$

where ω is an arbitrary formal product of homology cycles of X , and V_ω is a geometric representative of $\mu(\omega)$. Still more formally, one can write

$$[\mathcal{M}_{k+1} \cap \nu_1 \cap \nu_2] = 64[\mathcal{M}_k]. \quad (4)$$

I will show you that, with the right choice of ν_i , the boundary region of (a perturbation of) \mathcal{M}_{k+1} , intersected with ν_1 and ν_2 , looks like 6, not 64, copies of \mathcal{M}_k .

The Geometric Representative ν_p .

Our first step is to find appropriate geometric representatives of $\mu(x)$. The geometric representatives we will use are closely tied to the notion of reducibility. An $SU(2)$ connection is said to be reducible if the gauge group reduces to a proper subgroup of $SU(2)$. The only proper, nontrivial connected Lie subgroup of $SU(2)$ is $U(1)$, so the curvature of a reducible connection lives in the Lie Algebra of $U(1)$, which is 1-dimensional. Thus all the components of the curvature must be colinear. In fact, one can show that, on a contractible set, an $SU(2)$ connection is reducible if and only if, at each point, the components of the curvature are colinear. We therefore define “reducible at a point” to mean that the components of the curvature are colinear at that point.

Let p be any point in X , and let

$$\nu_p = \{[A] \in \mathcal{B}_k^* | F_A^- \text{ is reducible at } p\}. \quad (5)$$

Here $F_A^- = (F_A - *F_A)/2$ is the anti-self-dual part of the curvature F_A . In [S] I proved the following theorem, which applies equally well to $SU(2)$ and $SO(3)$ bundles.

Theorem 1: ν_p is a geometric representative of $-4\mu(x)$.

Sketch of Proof: We first need a geometric representative for the universal $SO(3)$ bundle, and then pull it back to a gauge theory setting. Let V be any real vector space, and let S_V be the Stiefel manifold of linearly independent triples of vectors in V . $SO(3)$ acts freely on S_V , with a quotient we denote G_V . Topologically, G_V is $\mathbb{R}^6 \times$ the Grassmannian of oriented 3-planes in V . If V is infinite-dimensional, then S_V is contractible, and $S_V \rightarrow G_V$ is the universal $SO(3)$ bundle. Let $\pi : V \rightarrow \mathbb{R}^3$ be any linear surjection. If $\dim(V) > 7$, then the first Pontryagin class of the bundle $S_V \rightarrow G_V$ is represented by

$$\nu_\pi = \{[v_1, v_2, v_3] \in G_V | \pi(v_1), \pi(v_2), \text{ and } \pi(v_3) \text{ are colinear.}\} \quad (6)$$

A proof, using Schubert cycles, may be found in [S], but this result was almost certainly known to Pontryagin.

We are now able to construct μ of the point class. Let p be a point on the manifold X , let D be a geodesic ball around p , let \mathcal{A}_D be the $SU(2)$ (or $SO(3)$) connections on D within the Sobolev space L_k^q (the choice of q and k is not important), let \mathcal{G}^0 be the gauge transformations in L_{k+1}^q that leave the fiber at p fixed, and let \mathcal{G} be all gauge transformations in L_{k+1}^q . Define $\mu_D(p)$ to be $-\frac{1}{4}p_1$ of the $SO(3)$ bundle $\mathcal{A}_D/\mathcal{G}^0 \rightarrow \mathcal{A}_D/\mathcal{G}$. $\mu_D(p)$ is a cohomology class in $H^*(\mathcal{B}(D))$. Let $R : \mathcal{B}(X) \rightarrow \mathcal{B}(D)$ be the map obtained by restricting connections on a bundle over X to a bundle over D , and define $\mu(x) = R^* \mu_D(p)$. $\mu(x)$ is then a class in $H^*(\mathcal{B}(X))$, which turns out not to depend on the choice of point p or neighborhood D . For more about the topology of the μ map, see Chapter 5 of [DK].

Note that, when the gauge group is $SU(2)$, the bundle $\mathcal{A}_D/\mathcal{G}^0 \rightarrow \mathcal{A}_D/\mathcal{G}$ is a principal $SO(3)$ bundle, not a principal $SU(2)$ bundle. The reason is that $SU(2)$ does not act freely on $\mathcal{A}_D/\mathcal{G}^0$. A gauge transformation by ± 1 leaves a connection fixed, so the typical fiber of our bundle is $SU(2)/Z_2 \sim SO(3)$.

Although the restriction of the original $SU(2)$ (or $SO(3)$) bundle to D is trivial, the bundle $\mathcal{A}_D/\mathcal{G}^0 \rightarrow \mathcal{A}_D/\mathcal{G}$ is highly nontrivial. Indeed, it is essentially the universal $SO(3)$

bundle. The space $\mathcal{A}_D/\mathcal{G}^0$ is isomorphic to the set of connections in radial gauge with respect to the point p . In such a gauge the connection form A vanishes in the radial direction but is otherwise unconstrained. In particular, $A(p) = 0$, and the curvature at p , $F_A(p) = dA(p) + A(p) \wedge A(p) = dA(p)$, is a linear function of A .

Let V be the space of (scalar valued) 1-forms with no radial component. A connection in radial gauge is defined by a triple of elements of V , one for each direction in the Lie Algebra. Deleting the infinite-codimension set for which these elements are linearly dependent we get S_V . Thus $\mu_D(p)$ is $-1/4p_1$ of $S_V \rightarrow G_V$, which we have already computed. Now let $\pi(\alpha)$ be $d^-(\alpha)$ evaluated at p . π is a linear map of V onto the 3-dimensional space of ASD 2-forms at p . $-4\mu_D(p)$ is then represented by ν_π , which is the set of connections on D for which $F^-(p)$ is reducible. Pulling $-4\mu_D(p)$ back by the restriction map we get the connections on X for which $F_A^-(p)$ is reducible, i.e. ν_p . ■

The Main Result.

Pick geodesic normal coordinates about some point in X , and let p and q be the points $(\pm L, 0, 0, 0)$, with L small. Our problem is to count the points on the left hand side of (3) that lie near the boundary of \mathcal{M}_{k+1} . This is tantamount to answering the basic question: *Given a connection $A_0 \in \mathcal{M}_k$, how many ways are there to glue a small charge-1 bubble onto A_0 so as to make the resulting curvatures reducible at both p and q ?* This is a local calculation, and yields an extremely simple answer:

Theorem 2: *For a generic background connection A_0 , and for any $\alpha \in (0, 2)$, there are exactly six ways to glue in a bubble of size $O(L^\alpha)$ so as to make $F^-(p)$ and $F^-(q)$ both reducible. All six solutions have bubbles of size $O(L^2)$, and all six have positive orientation. This answer is independent of the global topology and geometry of X , and in particular is independent of whether X has simple type.*

The Approximate Curvature Formula

Suppose we have a background connection A_0 , in a radial gauge with respect to the origin, and glue in a bubble with center at the origin, size λ , and gluing angle m to get a new connection A . What is the curvature F_A of A ? Remarkably, there is an extremely simple approximate formula:

$$F_A(x) \approx F_{A_0}(x) + F_{\text{std}}(x), \quad (7)$$

as long as $\lambda \ll |x| \ll 1$. Here F_{std} is the curvature of a standard $k = 1$ instanton, centered at the origin with size λ , in a gauge that is radial, singular at the origin, and regular at ∞ . This gauge is not unique; the choice of this gauge is essentially our gluing angle m . If a bubble is to be glued in at a point $y \neq 0$, then formula (7) still applies, except that the relevant gauges are radial with respect to y , not to the origin.

The reason for the formula is this. Let A_{std} be the connection form for the standard instanton. In the appropriate gauge, $|A_{\text{std}}(x)|$ is of order $\lambda^2/|x|^3$, while $|A_0(x)| = O(|x|)$. In the relevant region, the connection form for A is essentially $A_{\text{std}} + A_0$, and so the curvature is $F_0 + F_{\text{std}} + A_{\text{std}} \wedge A_0 + A_0 \wedge A_{\text{std}}$. The last two terms have norms of order $\lambda^2/|x|^2$, and so may be ignored for $|x| \gg \lambda$.

For the remainder of this paper, we will pretend that (7) is an equality, rather than an approximation. The error terms really do *not* matter, although it takes a fair bit of work [GS] to prove it.

Thanks to formula (7), our problem reduces to finding gluing data such that $F_0 + F_{\text{std}}$ is reducible at p and q . This involves two steps: finding what values of $F_{\text{std}}(p)$ and $F_{\text{std}}(q)$ are required, and counting the sets of gluing data that yield those values. Both steps require the following notational tool:

Expressing Curvatures as 3×3 Real Matrices

Relative to the standard oriented basis of $\Lambda_-^2 T^* \mathbb{R}^4$ (namely $\omega_1 = dx^0 dx^1 - dx^2 dx^3$, $\omega_2 = dx^0 dx^2 - dx^3 dx^1$, $\omega_3 = dx^0 dx^3 - dx^1 dx^2$), an anti-self-dual curvature form F has, at each point, 3 Lie-algebra-valued components. F can thus be viewed as a triple of 3-vectors, or equivalently a 3×3 real matrix that we (momentarily) denote $Mat(F)$. Reducibility at a point means that this matrix has a rank of 1 (or 0) there. More precisely, $Mat(F)$ is constructed as follows. The first, second and third columns of $Mat(F)$ are half the ω_1 , ω_2 and ω_3 components of F . The first, second and third entries of each column refer to the i , j and k directions in $su(2)$, the Lie Algebra of $SU(2)$. Of course, this construction is dependent on gauge and a choice of basis for TX . A gauge transformation is a change of basis in $su(2)$, and thus changes $Mat(F)$ by left-multiplication by an element of $SO(3)$. A change of basis in TX changes $Mat(F)$ by right-multiplication by an element of $SO(3)$. Thus the singular values of $Mat(F)$, and in particular the rank of $Mat(F)$, are gauge- and basis-independent.

Now let's compute the matrix of a standard $k = 1$ instanton. Think of $SU(2)$ as the unit quaternions, with $su(2)$ as the imaginary quaternions. The connection form of a standard instanton of scale size 1, centered at the origin, is $A_{\text{std}_0} = Im(\bar{x}dx/(1 + |x|^2))$. The curvature of this connection is

$$F_{\text{std}_0} = \frac{d\bar{x}dx}{(1 + |x|^2)^2} = \frac{2i\omega_1 + 2j\omega_2 + 2k\omega_3}{(1 + |x|^2)^2} \quad (8)$$

Note that the matrix $Mat(F_{\text{std}_0})$ is $1/(1 + |x|^2)^2$ times the identity matrix.

Unfortunately, that is in the wrong gauge, in which $A \sim \phi^{-1}d\phi$ as $|x| \rightarrow \infty$, where $\phi(x) = x/|x|$. We do a gauge transformation by ϕ^{-1} , to get a radial gauge in which $A = O(|x|^{-3})$ as $|x| \rightarrow \infty$ (and A is singular at the origin). We then do a further gauge transformation by an arbitrary constant g_0 to get the most general radial gauge with this property. F_{std} is the curvature form in this gauge. Since $F_{\text{std}} = g_0^{-1}\phi F_{\text{std}_0}\phi^{-1}g_0$, $Mat(F_{\text{std}}) = \rho(g_0^{-1})\rho(\phi)Mat(F_{\text{std}_0})$, where ρ is the standard double covering map from $SU(2)$ to $SO(3)$; the three columns of $\rho(\phi)$ are $\phi i \phi^{-1}$, $\phi j \phi^{-1}$, and $\phi k \phi^{-1}$. The matrix $\rho(g_0)$ is our gluing angle m .

Now suppose that we have a $k = 1$ instanton, centered at a point y , with scale size λ . The curvature matrix, expressed in the exterior radial gauge of gluing angle m , is then

$$Mat(F_{\text{std}}) = \frac{\lambda^2}{(\lambda^2 + |x - y|^2)^2} m^{-1} \rho \left(\frac{x - y}{|x - y|} \right) \quad (9)$$

Note that $Mat(F_{\text{std}})$ is a positive multiple of an $SO(3)$ matrix. The multiple is determined by λ and $|x - y|$, while the $SO(3)$ matrix is determined by m and $(x - y)/|x - y|$.

From now on, we will identify curvatures with their matrices, and will omit the explicit function “ Mat ”.

A Linear Algebra Lemma

Recall that the singular values $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$ of a 3×3 real matrix M are the square roots of the eigenvalues of $M^T M$. For M generic, these are distinct and positive. The non-generic cases are as follows: Matrices in a codimension-1 set have $\sigma_3 = 0$. Matrices in a codimension-2 set either have $\sigma_1 = \sigma_2$ or $\sigma_2 = \sigma_3$. Matrices in a codimension-4 set have $\sigma_2 = \sigma_3 = 0$; these matrices have rank 1 or 0. Matrices in a codimension-5 set have $\sigma_1 = \sigma_2 = \sigma_3$; these are all scalar multiples of $SO(3)$ matrices. Only the zero matrix (codimension-9) has $\sigma_1 = \sigma_2 = \sigma_3 = 0$.

Lemma: *Let P be a 3 by 3 real matrix with singular values $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$. If these singular values are all distinct, then there are exactly two pairs $(s, M) \in (0, \infty) \times SO(3)$ for which $P + sM$ has rank 1 (and no pairs (s, M) for which $P + sM = 0$). In both cases $s = \sigma_2(P)$. If exactly two of the singular values of P are the same and nonzero, then the two solutions (s, M) coalesce to a double root.*

The proof, although straightforward, is not especially enlightening, so we’ll skip it here. You may want to attempt it as an exercise, or even assign it to an advanced linear algebra class. One of the many possible proofs can be found in [GS].

This means that, for any generic background connection A_0 and a generic point z , there is one size s_z and two $SO(3)$ matrices $M_1(z)$ and $M_2(z)$ such that $F_0(z) + s_z M_i(z)$ is reducible. Note that these are completely determined by $F_0(z)$, which is a smooth function of z . Since p and q are separated by a distance $O(L)$, s_p is within $O(L)$ of s_q , and $M_i(p)$ is within $O(L)$ of $M_i(q)$.

Solving For The Magnitude Of F_{std}

According to formula (7), in order to have $F(p)$ and $F(q)$ reducible, we must solve $F_{\text{std}}(p) = s_p M_i(p)$ and $F_{\text{std}}(q) = s_q M_j(q)$. We do this in two steps, first solving for the magnitudes and then for the $SO(3)$ matrices. We will pretend that $s_p = s_q$. The actual $O(L)$ difference between them complicates the algebra, but does not make any qualitative difference.

The condition for the standard curvature F_{std} to have magnitude s_p at p is

$$\frac{\lambda^2}{(|y - p|^2 + \lambda^2)^2} = s_p, \quad (10)$$

or equivalently

$$\lambda^2 + |y - p|^2 = \lambda / \sqrt{s_p}. \quad (11)$$

Similarly, we need

$$\lambda^2 + |y - q|^2 = \lambda / \sqrt{s_q}. \quad (12)$$

Since $s_p = s_q$, this implies that $|y - p| = |y - q|$, so $y = (y_0, y_I)$ must lie in the plane half-way between p and q . That is, $y_0 = 0$. As long as this is the case, any solution to (11) is also a solution to (12).

As long as $|y - p| < 1/2\sqrt{s_p}$ there are two solutions to (11), while for $|y - p| > 1/2\sqrt{s_p}$ there are none. When $|y - p| < 1/2\sqrt{s_p}$, one solution has $\lambda > 1/2\sqrt{s_p}$, which is not $O(L^\alpha)$ for L small. The other solution may qualify as small if $|y - p|$ is small enough, and, for $|y - p| \ll 1/\sqrt{s_p}$, is approximately $\lambda = |y - p|^2\sqrt{s_p} = (L^2 + |y_I|^2)\sqrt{s_p}$.

Solving For The Gluing Angle

We still have to get the $SO(3)$ matrices right. By eq. (9), this means simultaneously solving the equations

$$m^{-1}\rho((y-p)/|y-p|) = M_i(p) \quad (13)$$

and

$$m^{-1}\rho((y-q)/|y-q|) = M_j(q) \quad (14)$$

for m . If a solution exists it is obviously unique. A solution exists if and only if $\rho((y-p)/|y-p|)^{-1}\rho((y-q)/|y-q|) = M_i(p)^{-1}M_j(q)$. Let

$$g(y) = \frac{(\bar{y} - \bar{p})(y - q)}{|(y - p)(y - q)|}. \quad (15)$$

We must count the points on our 3-disk (of small solutions to (11) and (12)) for which the $SO(3)$ -valued function $\rho(g(y))$ equals $M_i(p)^{-1}M_j(q)$. Note that

$$g(y) = -I + 2y_I/L + O((|y_I|/L)^2) \quad \text{for } |y_I| \ll L, \quad (16)$$

while

$$g(y) = I - 2Ly_I/|y_I|^2 + O((L/|y_I|)^2) \quad \text{for } |y_I| \gg L. \quad (17)$$

Pick a constant $K > 0$ and let $R_{K,\alpha}$ be such that $|y_I| < R_{K,\alpha}$ implies $\lambda \leq KL^\alpha$. For L small we have $R_{K,\alpha}^2 \sim KL^\alpha/\sqrt{s_m} - L^2 \sim KL^\alpha/\sqrt{s_m}$, since $\alpha < 2$. $L/R_{K,\alpha}$ is $O(L^{1-\alpha/2})$ and hence goes to zero as $L \rightarrow 0$. On the 3-disk of admissible y_I , the map g covers all of $SU(2)$ except for a ball of radius $\sim L^{1-\alpha/2}$ around the origin. Since ρ is a 2-1 map, $\rho(g(y))$ hits all of $SO(3)$ twice, except for a ball of radius $\sim L^{1-\alpha/2}$ around the origin, which is only hit once. The number of solutions to our problem depends on whether, for small L , $M_i^{-1}(p)M_j(q)$ is in this ball or not.

As $L \rightarrow 0$, $M_1(p)^{-1}M_2(q)$ and $M_2(p)^{-1}M_1(q)$ are bounded away from the identity, but $M_1(p)^{-1}M_1(q)$ and $M_2(p)^{-1}M_2(q)$ are within $O(L)$ (and hence within $o(L^{1-\alpha/2})$) of the identity. Thus we have two sets of parameters (y, λ, m) that give $F_{\text{std}}(p) = s_p M_1(p)$ and $F_{\text{std}}(q) = s_q M_2(q)$, two that give $F_{\text{std}}(p) = s_p M_2(p)$ and $F_{\text{std}}(q) = s_q M_1(q)$, one that gives $F_{\text{std}}(p) = s_p M_1(p)$ and $F_{\text{std}}(q) = s_q M_1(q)$ and one that gives $F_{\text{std}}(p) = s_p M_2(p)$ and $F_{\text{std}}(q) = s_q M_2(q)$. A total of six solutions in all.

Peeking Under The Carpet

If you accept all the simplifying assumptions I have made, then the proof of Theorem 2 is finished. I reduced the central question to counting the solutions to some explicit (and simple!) algebraic equations, and I not only counted the solutions, but actually showed you how to find them.

The skeptical among you, however, may be worried that my simplifying assumptions are unrealistic, or hide some deep problems. To ease your fears, here is a list of the shortcuts I have taken, and how these issues are dealt with in [GS].

I assumed that $s_p = s_q$ when, in fact, s_p and s_q differ by $O(L)$. The set of y for which both (11) and (12) can be solved for λ is typically not a disk in the plane $y_0 = 0$; rather, it is an ellipsoid that passes between p and q . The portion of the ellipsoid that results in λ being small is a topological 3-disk located between p and q . As $L \rightarrow 0$, this topological 3-disk approaches a geometric 3-disk in the plane $y_0 = 0$ sufficiently rapidly that the previous discussion goes through essentially unchanged.

I showed that there are 6 solutions, but did not show that they all give intersection number $+1$. This involves two steps. First we show that the intersection numbers are, by continuity, independent of $M_i(p)$ and $M_j(q)$. A similar argument shows that the intersection numbers for the two solutions for a given $M_i(p)$ and $M_j(q)$ are equal. We then compute the intersection number at $(y = 0, \lambda, m = I)$ for $M_i(p) = M_j(q) = I$, which is indeed $+1$.

In order to apply formula (7), A_0 must be in radial gauge with respect to y . However, we computed $M_i(p)$ and $M_j(q)$ from A_0 in radial gauge with respect to the origin, not with respect to y . In reality, $M_i(p)$ and $M_j(q)$ should really be viewed as functions of y . The extent of this y -dependence can be estimated, and we show that the derivative of $M_i(p)^{-1}M_j(q)$ with respect to y is too small to change the count.

Formula (7) is itself an approximation, not an equality. The error terms may be treated as a perturbation to F_0 . By estimating the dependence of (y, λ, m) on F_0 , and the dependence of the error terms of (y, λ, m) , we show that any solution to $F_0(p) + F_{\text{std}}(p) = \text{reducible}$ and $F_0(q) + F_{\text{std}}(q) = \text{reducible}$ can be perturbed to a solution to $F(p) = \text{reducible}$ and $F(q) = \text{reducible}$, and vice versa. These estimates are, technically, the most difficult part of the whole problem.

Finally, formula (7) applies not to a true ASD connection, but to a connection obtained by an explicit grafting formula. The set of such connections is an L^2 -small perturbation $\tilde{\mathcal{M}}_{k+1}$ of the true ASD moduli space \mathcal{M}_{k+1} . Theorem 2 does not directly relate \mathcal{M}_k to $\mathcal{M}_{k+1} \cap \nu_p \cap \nu_q$. Rather, it relates \mathcal{M}_k to $\tilde{\mathcal{M}}_{k+1} \cap \nu_p \cap \nu_q$.

Ideally, one would like to interpolate from $\tilde{\mathcal{M}}_{k+1} \cap \nu_p \cap \nu_q$ to $\mathcal{M}_{k+1} \cap \nu_p \cap \nu_q$. This is quite difficult, as ν_p and ν_q are defined by pointwise conditions. I know of no pointwise estimates relating the curvature of an almost-ASD connection to that of a nearby ASD connection. In order to make use of the integral estimates available in the literature one would have to replace ν_p and ν_q by geometric representatives defined by integral conditions. While this is possible (Cliff Taubes once showed me such an extended representative), it is well beyond the scope of this work.

A Differential Forms Approach

There is a quite different approach to measuring the importance of the boundary region of \mathcal{M}_{k+1} to simple type. One can use differential form representatives of $\mu(\cdot)$, and integrate these forms over \mathcal{M} to obtain Donaldson invariants. In that setting, our problem is to integrate $\mu_{dR}(p) \wedge \mu_{dR}(q)$ over the 8-dimensional space of gluing parameters. Here $\mu_{dR}(p)$ is a de Rham representative of $\mu(x)$ based on connections near p , much the way that ν_p is a geometric representative. Let B_{λ_0} be the set of gluing data (y, λ, m) for which $\lambda < \lambda_0$. In [GS] we prove that

$$\lim_{\lambda_0 \rightarrow 0} \lim_{L \rightarrow 0} \int_{B_{\lambda_0}} \mu_{dR}(p) \wedge \mu_{dR}(q) = 1/2. \quad (18)$$

Surprisingly, this is a different answer than obtained from the geometric representative calculation (1/8 of what is required for simple type, as opposed to 6/64). Moreover, the bulk of the integral (18) is from λ being of order L , while the geometric representative calculation had all the intersection points having λ of order L^2 . This is not a contradiction. Although the Donaldson invariants are topological, hence independent of a choice of representatives, the contribution of the boundary region is geometric, and can definitely depend on a choice of representatives.

Conclusions

While the two approaches disagree on the exact contribution of the boundary region, and on just how close to the boundary we should consider, they agree on the central theme of this paper. Simple type is *not* a boundary phenomenon. Simple type implies that the features of each moduli space \mathcal{M}_k are duplicated in the structure of the interior of \mathcal{M}_{k+1} . This duplication is not at all explained by our present understanding of moduli spaces; perhaps the explanation lies in quantum duality.

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